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Research on Some Fractional Integral Problems

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Abstract: **In this paper, we solve two type of fractional integrals based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus. The main two methods used in this article are change of variables for fractional integral and integration by parts for fractional calculus. On the other hand, a new multiplication of fractional analytic functions plays an important role in this paper. And these two types of fractional integrals are generalizations of the integrals in traditional calculus.**

Keyword: **fractional integrals, Jumarie's modified R-L fractional calculus, change of variables, integration by parts, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining real order or complex order. In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. Fractional calculus is widely welcomed and concerned because of its applications in many fields such as mechanics, dynamics, modelling, physics, economics, viscoelasticity, biology, electronics, signal processing, and so on [1-9]. However, fractional calculus is different from ordinary calculus. The definition of fractional derivative and integral is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [10-13].

In this paper, we study the following two type of fractional integrals:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes m}\otimes\left(E_{\alpha}(x^{\alpha})\right)^{\otimes n}\right], \tag{1}$$

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes p}\otimes\left(Ln_{\alpha}\left(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right)^{\otimes q}\right]. \tag{2}$$

Where $0<\alpha\leq 1$, and m,n,p,q are positive integers. The change of variables for fractional integral and the integration by parts for fractional calculus are the main methods used in this article. In addition, a new multiplication of fractional analytic functions plays an important role in this paper. In fact, the above two types of fractional integrals are generalizations of the integrals in classical calculus.

II. PRELIMINARIES

First, the fractional calculus used in this paper and its properties are introduced below.

Definition 2.1 ([14])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{3}$$

And the Jumarie's modified R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{4}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([15])**:** *If* α,β,x_0,C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{6}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([16])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([17])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{7}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{8}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{9}$$

In other words,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{10}$$

Definition 2.5 ([18])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{11}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{12}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{13}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\big(f_\alpha(x^\alpha)\big)^{\otimes k}. \tag{14}$$

Definition 2.6 ([18])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{15}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7: Suppose that n is a positive integer, then $\big(f_\alpha(x^\alpha)\big)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the nth power of the fractional analytic function $f_\alpha(x^\alpha)$.

Theorem 2.8 (change of variables for fractional integral) ([19])**:** *If $0 < \alpha \le 1$, g_α is α-fractional analytic at $x = a$, f_α is α-fractional analytic at $x = g_\alpha(a)$ and the range of g_α contained in the domain of f_α, then $f_\alpha \circ g_\alpha$ is α-fractional analytic at $x = a$, and*

$$\left({}_aI_b^\alpha\right)\left[(f_\alpha \circ g_\alpha)(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = \left({}_{g_\alpha(a)}I_{g_\alpha(b)}^\alpha\right)[f_\alpha(g_\alpha)], \tag{16}$$

for $a, b \in I$.

Theorem 2.9 (integration by parts for fractional calculus) ([20])**:** *Suppose that $0 < \alpha \le 1$, a, b are real numbers, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$* are α-fractional analytic functions, *then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = \left[\, f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)\right]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{17}$$

~~tnote reference:~~Some fractional analytic functions are introduced below.

Definition 2.10 ([18])**:** If $0 < \alpha \le 1$, and x, x_0 are real numbers. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{18}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

III. MAIN RESULTS

The followings are major results in this paper.

Theorem 3.1: *Let $0 < \alpha \le 1$, and m, n be positive integers. Then the α-fractional integral*

$$\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes m} \otimes \big(E_\alpha(x^\alpha)\big)^{\otimes n}\right]$$

$$= \left[\sum_{k=0}^{m} \frac{(-1)^k \cdot (m)_k}{n^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(m-k)}\right] \otimes E_\alpha(nx^\alpha) - \frac{(-1)^m \cdot m!}{n^{m+1}}. \tag{19}$$

Where $(m)_k = m(m-1)\cdots(m-k+1)$ for positive integers k, and $(m)_0 = 1$.

Proof Using integration by parts for fractional calculus yields

$$\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes m} \otimes \big(E_\alpha(x^\alpha)\big)^{\otimes n}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes m} \otimes E_\alpha(nx^\alpha)\right]$$

$$= \frac{1}{n}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes m} \otimes E_\alpha(nx^\alpha) - \frac{m}{n^2}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(m-1)} \otimes E_\alpha(nx^\alpha) + \cdots$$

$$+ \frac{(-1)^m \cdot m!}{n^m}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(m-1)} \otimes E_\alpha(nx^\alpha) + \frac{(-1)^m \cdot m!}{n^{m+1}} \otimes E_\alpha(nx^\alpha) - \frac{(-1)^m \cdot m!}{n^{m+1}}$$

$$= \left[\sum_{k=0}^{m} \frac{(-1)^k \cdot (m)_k}{n^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(m-k)}\right] \otimes E_\alpha(nx^\alpha) - \frac{(-1)^m \cdot m!}{n^{m+1}}.$$ Q.e.d.

Theorem 3.2: *If* $0 < \alpha \leq 1$, *and* p, q *are positive integers. Then the* α*-fractional integral*

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes p}\otimes\left(Ln_{\alpha}(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha})\right)^{\otimes q}\right]$$

$$=\sum_{j=0}^{p}\sum_{k=0}^{q}\frac{(-1)^{j+k}\binom{p}{j}\cdot(q)_{k}}{(p+1-j)^{k+1}}\left(Ln_{\alpha}(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha})\right)^{\otimes(q-k)}\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(p+1-j)}-\sum_{j=0}^{p}\frac{(-1)^{q+j}\cdot\binom{p}{j}\cdot q!}{(p+1-j)^{q+1}}. \quad (20)$$

Proof Let $1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}=E_{\alpha}(t^{\alpha})$, then $Ln_{\alpha}(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha})=\frac{1}{\Gamma(\alpha+1)}t^{\alpha}$. By change of variables for fractional integral and Theorem 3.1, we obtain

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes p}\otimes\left(Ln_{\alpha}(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha})\right)^{\otimes q}\right]$$

$$=\left({}_{0}I_{t}^{\alpha}\right)\left[E_{\alpha}(t^{\alpha})\otimes(E_{\alpha}(t^{\alpha})-1)^{\otimes p}\otimes\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes q}\right]$$

$$=\left({}_{0}I_{t}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes q}\otimes\left[\left(E_{\alpha}(t^{\alpha})\right)^{\otimes(p+1)}-\binom{p}{1}\left(E_{\alpha}(t^{\alpha})\right)^{\otimes p}+\binom{p}{2}\left(E_{\alpha}(t^{\alpha})\right)^{\otimes(p-1)}-\cdots+(-1)^{p}\binom{p}{p}\left(E_{\alpha}(t^{\alpha})\right)^{\otimes 1}\right]\right]$$

$$=\left({}_{0}I_{t}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes q}\otimes\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}E_{\alpha}((p+1-j)t^{\alpha})\right]$$

$$=\left({}_{0}I_{t}^{\alpha}\right)\left[\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes q}\otimes E_{\alpha}((p+1-j)t^{\alpha})\right]$$

$$=\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}\left({}_{0}I_{t}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes q}\otimes E_{\alpha}((p+1-j)t^{\alpha})\right]$$

$$=\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}\left(\left[\sum_{k=0}^{q}\frac{(-1)^{k}\cdot(q)_{k}}{(p+1-j)^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(q-k)}\right]\otimes E_{\alpha}((p+1-j)t^{\alpha})\right)-\sum_{j=0}^{p}\frac{(-1)^{q+j}\cdot\binom{p}{j}\cdot q!}{(p+1-j)^{q+1}}$$

$$=\sum_{j=0}^{p}\sum_{k=0}^{q}\frac{(-1)^{j+k}\binom{p}{j}\cdot(q)_{k}}{(p+1-j)^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(q-k)}\otimes E_{\alpha}((p+1-j)t^{\alpha})-\sum_{j=0}^{p}\frac{(-1)^{q+j}\cdot\binom{p}{j}\cdot q!}{(p+1-j)^{q+1}}$$

$$=\sum_{j=0}^{p}\sum_{k=0}^{q}\frac{(-1)^{j+k}\binom{p}{j}\cdot(q)_{k}}{(p+1-j)^{k+1}}\left(Ln_{\alpha}(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha})\right)^{\otimes(q-k)}\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(p+1-j)}-\sum_{j=0}^{p}\frac{(-1)^{q+j}\cdot\binom{p}{j}\cdot q!}{(p+1-j)^{q+1}}.$$

Q.e.d.

IV. CONCLUSION

This paper studies two type of fractional integrals based on Jumarie type of R-L fractional calculus. The major methods we used are change of variables for fractional integral and integration by parts for fractional calculus. A new multiplication of fractional analytic functions plays an important role in this article. In fact, these two types of fractional integrals are generalizations of integrals in ordinary calculus. In the future, we will continue to use these methods to study problems in fractional calculus and applied mathematics.

REFERENCES

[1] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics, vol.23, pp. 397-404. 2002.

[2] R. L. Magin, Modeling the cardiac tissue electrode interface using fractional calculus, Journal of Vibration and Control, vol. 14, no. 9-10, pp. 1431-1442, 2008.

[3] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[4] F. B. Adda and J. Cresson, Fractional differential equations and the Schrödinger equation, Applied Mathematics and Computation, vol. 161, pp. 323-345, 2005.

[5] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, vol. 32, no,1. pp.1-104, 2004.

[6] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[7] C. -H. Yu, Study on fractional Newton's law of cooling, International Journal of Mechanical and Industrial Technology, vol. 9, no. 1, pp. 1-6, 2021.

[8] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[9] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[10] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[11] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[13] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[14] C. -H. Yu, Fractional derivative of arbitrary real power of fractional analytic function, International Journal of Novel Research in Engineering and Science, vol. 9, no. 1, pp. 9-13, 2022.

[15] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[16] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[17] C. -H. Yu, A study on arc length of nondifferentiable curves, Research Inventy: International Journal of Engineering and Science, vol. 12, no. 4, pp. 18-23, 2022.

[18] C. -H. Yu, Using trigonometric substitution method to solve some fractional integral problems, International Journal of Recent Research in Mathematics Computer Science and Information Technology, vol. 9, no. 1, pp. 10-15, 2022.

[19] C. -H. Yu, Some techniques for evaluating fractional integrals, 2021 International Conference on Computer, Communication, Control, Automation and Robotics, Journal of Physics: Conference Series, IOP Publishing, vol. 1976, 012081, 2021.

[20] C. -H. Yu, Some applications of integration by parts for fractional calculus, International Journal of Computer Science and Information Technology Research, vol.10, no. 1, pp. 38-42, 2022.